

# Semigroup cohomology as a derived functor

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## Abstract

In this work we construct an extension for the category of 0-modules by analogy with [5]. The 0-cohomology functor becomes a derived functor in the extended category. As an application of this construction we calculate the cohomological dimension of so-called 0-free monoids.

1. 0-cohomology of semigroups appeared in research of projective representations of semigroups [1]. Besides, it was useful in studying of matrix algebras [3] and Brauer monoids [4] (see also survey [2] and references there).

However the further study of its properties is complicated. One of the reasons is that the semigroup 0-cohomology is not a derived functor in the category where it is built (so-called category of 0-modules).

The purpose of this paper is to describe the extension of 0-cohomology on a larger category where it becomes a derived functor. Our construction is similar to Baues theory for cohomology of small categories [5]. Therefore we omit some proofs replacing them by references to [5].

As an example of application of our construction we prove that a cohomological dimension of a so-called 0-free semigroup equals one. In particular, it follows that all projective representations of a free semigroup are linearizable.

2. We begin with definitions. Let  $S$  be a monoid. We may assume that  $S$  has a zero element (if not, let us join it to  $S$ ). By analogy with [5] *the category of factorizations in  $S$*  is given as follows. The objects are all nonzero elements of  $S$  and the set of morphisms  $\text{Mor}(a, b)$  consists of all triples  $(\alpha, a, \beta)$  ( $\alpha, \beta \in S$ ) such that  $\alpha a \beta = b$ . We will denote  $(\alpha, a, \beta)$  by  $(\alpha, \beta)$  if this cannot lead to confusion. The composition is defined by the rule:  $(\alpha', \beta')(\alpha, \beta) = (\alpha' \alpha, \beta \beta')$ ; hence we have  $(\alpha, \beta) = (\alpha, 1)(1, \beta) = (1, \beta)(\alpha, 1)$ . Denote this category by  $\mathcal{Fac}S$ .

A *natural system on  $S$*  is a functor  $\mathbf{D} : \mathcal{Fac}S \rightarrow \mathcal{Ab}$ . The category  $\mathcal{Nat}S = \mathcal{Ab}^{\mathcal{Fac}S}$  is an Abelian category with enough projectives and injectives [6]. Denote the value of  $\mathbf{D}$  at the object  $a \in \text{Ob}\mathcal{Fac}S$  by  $\mathbf{D}_a$ . By  $\alpha_*$

and  $\beta^*$  denote values of  $\mathbf{D}$  at morphisms  $(\alpha, 1)$  and  $(1, \beta)$  respectively. We have  $\mathbf{D}(\alpha, \beta) = \alpha_* \beta^*$  for all morphisms  $(\alpha, \beta)$ .

For given natural number  $n$  denote by  $Ner_n S$  the set of all  $n$ -tuples  $(a_1, \dots, a_n)$ ,  $a_i \in S$ , such that  $a_1 \cdots a_n \neq 0$ . By definition  $Ner_0 S = \{1\}$ . A  $n$ -cochain assigns to each point  $a = (a_1, \dots, a_n)$  of  $Ner_n S$  an element on  $\mathbf{D}_{a_1 \cdots a_n}$ . The set of all  $n$ -cochains is an Abelian group  $C^n(S, \mathbf{D})$  with respect to the pointwise addition. Set  $C^0(S, \mathbf{D}) = \mathbf{D}_1$ .

The coboundary  $\delta = \delta^n : C^n(S, \mathbf{D}) \longrightarrow C^{n+1}(S, \mathbf{D})$  is given by the formula ( $n \geq 1$ )

$$\begin{aligned} (\delta f)(a_1, \dots, a_{n+1}) &= a_1 * f(a_2, \dots, a_{n+1}) \\ &+ \sum_{i=1}^n (-1)^i f(a_1, \dots, a_i a_{i+1}, \dots, a_{n+1}) + (-1)^{n+1} a_{n+1}^* f(a_1, \dots, a_n). \end{aligned}$$

For  $n = 0$  let  $\delta^0 : C^0(S, \mathbf{D}) \longrightarrow C^1(S, \mathbf{D})$  be defined by

$$\delta f(x) = x_* f - x^* f \quad (f \in D_1, \quad x \in S \setminus 0).$$

One can check directly that  $\delta^n \delta^{n-1} = 0$ . By  $H^n(S, \mathbf{D})$  denote the cohomology groups of the complex  $\{C^n(S, \mathbf{D}), \delta^n\}_{n \geq 0}$ .

**3.** Now we define a *trivial natural system*  $\mathbf{Z}$ . To each object  $a \in S \setminus 0$  it assigns the infinite cyclic group  $\mathbf{Z}_a$  generated by a symbol  $[a]$ ; and to each morphism  $(\alpha, \beta) : a \longrightarrow b$  it assigns a homomorphism of the groups  $\mathbf{Z}(\alpha, \beta) : \mathbf{Z}_a \longrightarrow \mathbf{Z}_b$  which takes  $[a]$  to  $[b]$ .

Since  $\mathcal{Nat} S$  has enough projective and injective, hence there exists the derived functor  $\text{Ext}_{\mathcal{Nat} S}^n(\mathbf{Z}, -)$ . This functor is isomorphic to the cohomology functor  $H^n(S, -)$  which is defined in Section 2. To prove this statement we construct a suitable projective resolution of  $\mathbf{Z}$ .

For every  $n \geq 0$  we denote by  $\mathbf{B}_n : \mathcal{Fac} S \longrightarrow \mathcal{Ab}$  the following natural system. For an object  $a \in S \setminus 0$  the group  $\mathbf{B}_n(a)$  is a free Abelian group generated by the set of symbols  $[a_0, \dots, a_{n+1}]$  such that  $a_0 \cdots a_{n+1} = a$ . To each morphism  $(\alpha, \beta)$  we assign a homomorphism of groups by the formula

$$\mathbf{B}_n(\alpha, \beta) : [a_0, \dots, a_{n+1}] \longmapsto [\alpha a_0, \dots, a_{n+1} \beta].$$

The functors  $\mathbf{B}_n$  ( $n \geq 0$ ) constitute a chain complex  $\{\mathbf{B}_n, \partial_n\}_{n \geq 0}$ , where  $\partial_n : \mathbf{B}_n \longrightarrow \mathbf{B}_{n-1}$  ( $n \geq 1$ ) is a natural transformation with the set of its components

$$(\partial_n)_a : \mathbf{B}_n(a) \longrightarrow \mathbf{B}_{n-1}(a),$$

$$(\partial_n)_a[a_0, \dots, a_{n+1}] = \sum_{i=0}^n (-1)^i [a_0, \dots, a_i a_{i+1}, \dots, a_{n+1}].$$

**4. LEMMA.** *The natural system  $\mathbf{B}_n$  is a projective object in  $\mathcal{N}atS$ .*

PROOF. Consider the following diagram with the exact row

$$\begin{array}{ccc} & \mathbf{B}_n & \\ & \downarrow \nu & \\ \mathbf{D} & \xrightarrow{\mu} \mathbf{E} & \longrightarrow 0 \end{array}$$

and construct a natural transformation  $\tau : \mathbf{B}_n \dashrightarrow \mathbf{D}$  which turns this diagram into commutative.

Let  $s = s_0 \cdots s_{n+1}$ ,  $\hat{s} = s_1 \cdots s_n$ . Choose  $a_{(s_1, \dots, s_n)} \in \mathbf{D}(\hat{s})$  such that  $\mu_{\hat{s}} a_{(s_1, \dots, s_n)} = \nu_{\hat{s}}[1, s_1, \dots, s_n, 1]$ , and put

$$\tau_s[s_0, \dots, s_{n+1}] = \mathbf{D}(s_0, s_{n+1})a_{(s_1, \dots, s_n)}.$$

The natural transformation is well defined. Indeed,

$$\begin{aligned} \tau_{\alpha s \beta} \mathbf{B}_n(\alpha, \beta)[s_0, \dots, s_{n+1}] &= \mathbf{D}(\alpha s_0, s_{n+1} \beta) a_{(s_1, \dots, s_n)} = \\ &= \mathbf{D}(\alpha, \beta) \mathbf{D}(s_0, s_{n+1}) a_{(s_1, \dots, s_n)} = \mathbf{D}(\alpha, \beta) \tau_s[s_0, \dots, s_{n+1}]. \quad \square \end{aligned}$$

**5. LEMMA.** *The chain complex  $\{\mathbf{B}_n, \partial_n\}_{n \geq 0}$  is a projective resolution of the natural system  $\mathbf{Z}$ .*

The proof is similar to [5].

**6.** Now we are ready to prove the main result of this paper.

**THEOREM.** *For any monoid  $S$  with a zero element there is an isomorphism of the functors:*

$$H^n(S, -) \cong \text{Ext}_{\mathcal{N}atS}^n(\mathbf{Z}, -).$$

PROOF. Define an isomorphism of complexes

$$\Psi_{\mathbf{D}}^* : \{\text{Hom}_{\mathcal{N}atS}(\mathbf{B}_n, \mathbf{D}), \partial^n\}_{n \geq 0} \longrightarrow \{C^n(S, \mathbf{D}), \delta^n\}_{n \geq 0}$$

(here we denote  $\partial^n = \text{Hom}_{\mathcal{N}atS}(\partial_{n-1}, \mathbf{D})$ ) as follows. Let the homomorphism of Abelian group

$$\Psi_{\mathbf{D}}^n : \text{Hom}_{\mathcal{N}atS}(\mathbf{B}_n, \mathbf{D}) \longrightarrow C^n(S, \mathbf{D})$$

be given by

$$(\Psi_{\mathbf{D}}^n \tau)(a_1, \dots, a_n) = \tau_{a_1 \dots a_n}[1, a_1, \dots, a_n, 1] \in \mathbf{D}_{a_1 \dots a_n} \text{ for } a_1 \dots a_n \neq 0.$$

Let  $a = a_0 \dots a_{n+1}$ , i.e.  $[a_0, \dots, a_{n+1}] \in \mathbf{B}_n(a)$ . Since the diagram

$$\begin{array}{ccc} \mathbf{B}_n(a_1 \dots a_n) & \xrightarrow{\tau_{a_1 \dots a_n}} & \mathbf{D}_n(a_1 \dots a_n) \\ \mathbf{B}_n(a_0, a_{n+1}) \downarrow & & \downarrow \mathbf{D}_n(a_0, a_{n+1}) \\ \mathbf{B}_n(a) & \xrightarrow{\tau_a} & \mathbf{D}_n(a) \end{array}$$

is commutative we have

$$\tau_a[a_0, \dots, a_{n+1}] = \mathbf{D}(a_0, a_{n+1})\tau_{a_1 \dots a_n}[1, a_1, \dots, a_n, 1].$$

Therefore  $\Psi_{\mathbf{D}}^n \tau = 0$  implies that  $\tau_a$  vanishes on all generators of the group  $\mathbf{B}_n(a)$ . Hence  $\Psi_{\mathbf{D}}^n$  is injective.

Further, for any  $f \in C^n(S, \mathbf{D})$  define a natural transformation  $\varphi : \mathbf{B}_n \rightarrow \mathbf{D}$ :

$$\varphi_a[a_0, \dots, a_{n+1}] = \mathbf{D}(a_0, a_{n+1})f(a_1, \dots, a_n)$$

It is clear that  $\Psi_{\mathbf{D}}^n \varphi = f$  and hence  $\Psi^n$  is surjective. The commutativity of the diagram

$$\begin{array}{ccc} \text{Hom}_{\mathcal{Nat}S}(\mathbf{B}_n, \mathbf{D}) & \xrightarrow{\partial^n} & \text{Hom}_{\mathcal{Nat}S}(\mathbf{B}_{n+1}, \mathbf{D}) \\ \Psi_{\mathbf{D}}^n \downarrow & & \downarrow \Psi_{\mathbf{D}}^{n+1} \\ C^n(S, \mathbf{D}) & \xrightarrow{\delta^n} & C^{n+1}(S, \mathbf{D}) \end{array}$$

is established immediately.

It can easily be checked that the family  $\Psi^n = \{\Psi_{\mathbf{D}}^n | \mathbf{D} \in \mathcal{Nat}S\}$  is a natural transformation. From above we see that  $\Psi^n$  induces an isomorphism of functors  $H^n$  and  $\text{Ext}^n$ .  $\square$

**7.** Let us discuss the relation between cohomology which is defined above and cohomology groups of other kinds. In Section 1 we note that the 0-cohomology is a particular case of our construction. This can be shown in the following way. Let  $A$  be an Abelian group and  $\mathbf{A}$  be a natural system given by

$$\mathbf{A}(s) = A \text{ and } \alpha_* \beta^* a = \alpha a$$

for all  $s \in \mathcal{F}acS$ ,  $(\alpha, \beta) \in \text{Mor}\mathcal{F}acS$ . In other words,  $\mathbf{A}$  is so-called 0-module over  $S$  [1]: an action  $(S \setminus \{0\}) \times A \longrightarrow A$  is given, which satisfies the following conditions:

$$s(a + b) = sa + sb,$$

$$st \neq 0 \Rightarrow s(ta) = (st)a,$$

where  $s, t \in S \setminus 0$  and  $a, b \in A$ . 0-Cohomology groups are denoted by  $H_0^n(S, A)$ .

Note that Eilenberg-MacLane cohomology of semigroups [8] can be considered as a particular case of the 0-cohomology. Namely, if  $S$  is a semigroup (possibly without a zero), then  $H^n(S, -) \cong H_0^n(S^0, -)$ , where  $S^0$  is the semigroup  $S$  with an adjoint zero.

The category of 0-modules arises naturally in applications of 0-cohomology theory [4]. However it is easily shown that the second 0-cohomology group of the commutative semigroup  $S = \{u, v, w, 0\}$  with  $u^2 = v^2 = uv = w$ ,  $uw = vw = 0$  (see [1]) is not trivial for all nonzero 0-module over  $S$ . Therefore the 0-cohomology is not a derived functor on the category of 0-modules. This is the reason for introducing the category  $\mathcal{N}atS$ .

Our construction differs from Baues' cohomology theory for monoids [5] in the first step only. Actually in [5] a monoid  $S$  is regarded as a category with a single object. At the same time the Baues' category of factorizations in  $S$  is equal to  $\mathcal{F}acS^0$  out of Section 2. Therefore the Baues' cohomology groups of monoid  $S$  and cohomology groups of  $S^0$  in our sense are the same. However if  $S$  possesses a zero element then the category  $\mathcal{F}acS$  and Baues' one are not equivalent and we obtain the different cohomology groups.

**8.** Let us consider an application of the obtained results. *Cohomological dimension*  $c.d.S$  of monoid  $S$  is the greatest natural number such that  $H^n(S, \mathbf{D}) \neq 0$  for some  $\mathbf{D} \in \mathcal{N}atS$ . The Theorem from Section 6 allows us to use a projective resolution for calculation of the dimension.

It is well-known that in many cohomological theories  $c.d.$  of free objects equals 1. Free objects in the class of monoids with zero are free monoids with adjoint zero element. Nevertheless in our case the family of monoids having  $c.d.1$  is larger.

A monoid is called a *0-free monoid* if it is isomorphic to a Rees factor monoid of a free monoid. Free monoids with adjoint zero will be regarded as 0-free monoids too.

**9.** We shall need the following

LEMMA. Let  $\mathcal{A}, \mathcal{B}$  be categories,  $\mathbf{F} : \mathcal{A} \longrightarrow \mathcal{B}$ ,  $\mathbf{G} : \mathcal{B} \longrightarrow \mathcal{A}$  be adjoint

functors  $(\mathbf{F} \dashv \mathbf{G})$ , functor  $\mathbf{G}$  preserves epimorphisms and the counit  $\varepsilon : \mathbf{F}\mathbf{G} \xrightarrow{\cdot} \text{Id}_{\mathcal{B}}$  is identical. If an object  $a \in \mathcal{A}$  is projective then  $\mathbf{F}(a)$  is projective too.

PROOF. Let  $a \in \mathcal{A}$  be a projective object. Consider a diagram

$$\begin{array}{ccc} & \mathbf{F}(a) & \\ & \downarrow \alpha & \\ c & \xrightarrow{\beta} & b \end{array}$$

with the exact row  $(c, b \in \mathcal{B})$ . Since functor  $\mathbf{G}$  preserves epimorphisms we obtain the diagram:

$$\begin{array}{ccc} & a & \\ & \downarrow \mathbf{G}(\alpha)\eta_a & \\ \mathbf{G}(c) & \xrightarrow{\mathbf{G}(\beta)} & \mathbf{G}(b) \end{array} \quad (1)$$

where  $\eta : \text{Id}_{\mathcal{A}} \xrightarrow{\cdot} \mathbf{G}\mathbf{F}$  is the unit of the adjunction  $\mathbf{F} \dashv \mathbf{G}$ . Since  $a$  is projective, there is a homomorphism  $\gamma : a \rightarrow \mathbf{G}(c)$  which makes diagram (1) commutative. This means that  $\mathbf{G}(\beta)\gamma = \mathbf{G}(\alpha)\eta_a$  and  $\beta\mathbf{F}\gamma = \alpha\mathbf{F}(\eta_a)$ . Using the equalities  $\mathbf{F}(\eta_a) = \text{Id}_{\mathbf{F}(a)}$  and  $\mathbf{F}\mathbf{G} = \text{Id}_{\mathcal{B}}$  we get  $\beta\mathbf{F}\gamma = \alpha$ .  $\square$

**10. THEOREM.** *c.d.*  $M \leq 1$  for all 0-free monoids  $M$ .

PROOF. For a given monoid  $M$  consider the exact sequence

$$0 \longrightarrow \mathbf{P}_M \xrightarrow{\cdot} \mathbf{B}_M \xrightarrow{\cdot} \mathbf{Z}_M \longrightarrow 0$$

where  $\mathbf{Z}_M, \mathbf{B}_M$  are natural systems defined in Section 3,  $\mathbf{P}_M = \text{Ker}(\mathbf{B}_M \xrightarrow{\cdot} \mathbf{Z}_M)$ . We need to prove that  $\mathbf{P}_M$  is a projective functor.

It follows from Section 7 that  $\mathbf{P}_M$  is a free functor whenever  $M$  is a free monoid with adjoint zero (see [5], Lemma 6.7).

Now let  $M$  be a 0-free monoid,  $M \cong W/I$  where  $W$  is a free monoid and  $I$  is an ideal in  $W$ . Consider the category of factorizations  $\mathbf{F}W$  which was defined in [5], i.e.  $\mathbf{F}W = \mathcal{F}ac(W^0)$ . Define the functor  $\mathbf{K} : \mathcal{F}acM \rightarrow \mathbf{F}W$  which takes each nonzero element from  $M$  to its preimage under the canonic homomorphism  $W \rightarrow W/I$ . Functor  $\mathbf{K}$  is well defined and induces the functor  $\mathbf{K}^* : \mathbf{Nat}W \rightarrow \mathcal{N}atM$ , where  $\mathbf{Nat}W = \mathcal{A}b^{\mathbf{F}W}$ .

Consider the exact sequence which is defined in [5], Sec.5:

$$0 \longrightarrow \tilde{\mathbf{P}}_W \xrightarrow{\tilde{\delta}_W} \tilde{\mathbf{B}}_W \xrightarrow{\tilde{\varepsilon}_W} \tilde{\mathbf{Z}}_W \longrightarrow 0,$$

where  $\tilde{\mathbf{P}}_W, \tilde{\mathbf{B}}_W, \tilde{\mathbf{Z}}_W : \mathbf{F}W \longrightarrow \mathcal{A}b$  are natural systems on  $W$ . We have

$$\mathbf{K}^*(\tilde{\mathbf{Z}}_W) = \mathbf{Z}_M, \mathbf{K}^*(\tilde{\mathbf{B}}_W) = \mathbf{B}_M, \mathbf{K}^*(\tilde{\varepsilon}_W) = \varepsilon_M$$

hence  $\mathbf{K}^*(\tilde{\mathbf{P}}_W) = \mathbf{P}_M$ .

Consider the functor  $\mathbf{L} : \mathcal{N}atM \longrightarrow \mathbf{Nat}W$  which is given by

$$\mathbf{L}(\mathbf{G})_a = \begin{cases} \mathbf{G}_a, & \text{if } a \notin I \\ 0, & \text{if } a \in I \end{cases}$$

where  $\mathbf{G} \in \mathcal{N}atM$ , and

$$\mathbf{L}(\mathbf{G})(x, a, y) = \begin{cases} \mathbf{G}(x, a, y), & \text{if } xay \notin I \\ 0, & \text{if } xay \in I \end{cases}$$

Evidently  $\mathbf{K}^*\mathbf{L} = \text{Id}_{\mathcal{N}atM}$  and there is a natural transformation  $\text{Id}_{\mathbf{Nat}W} \xrightarrow{\cdot} \mathbf{L}\mathbf{K}^*$ . It implies that  $\mathbf{L}$  is right adjoint to  $\mathbf{K}^*$ . Besides,  $\mathbf{L}$  preserves epimorphisms and by [5]  $\tilde{\mathbf{P}}_W$  is a free object. Using Lemma 9 we get  $\mathbf{P}_M$  is a projective object.  $\square$

**11.** The semigroup is called *0-cancellative* if

$$ax = bx \neq 0 \Rightarrow a = b \text{ and } xa = xb \neq 0 \Rightarrow a = b$$

for all elements  $a, b, x$ . In view of Theorem 10 the following question arises: is a 0-cancellative monoid of cohomological dimension one a 0-free monoid?

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